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Speedy Motions of a Body Immersed in an Infinitely Extended Medium

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Abstract We study the motion of a classical point body of mass M, moving under the action of a constant force of intensity E and immersed in a Vlasov fluid of free particles, interacting with the body via a bounded short range potential Ψ. We prove that if its initial velocity is large enough then the body escapes to infinity increasing its speed without any bound (*runaway effect*). Moreover, the body asymptotically reaches a uniformly accelerated motion with acceleration E/M. We then discuss at a heuristic level the case in which $\Psi(r)$ diverges at short distances like $gr^{-\alpha}$, $g, \alpha > 0$, by showing that the runaway effect still occurs if $\alpha < 2$.

Keywords Viscous friction · Runaway particle · Vlasov fluid · Hamiltonian system

1 Introduction

In the present paper we study the longtime behavior of a classical point body moving under the action of a constant force $\boldsymbol{E}$ and immersed in an infinitely extended medium. This problem has been largely studied in the framework of kinetic theories, while here we want to obtain some rigorous results in the case of a fully Hamiltonian system.

Of course, when the medium is absent the body performs an uniformly accelerated motion. In the presence of a medium the motion is much more complicated and its asymptotic behavior depends on the nature of the background. In general, when the medium is homogeneous two different effects can happen: for strong body/medium interactions the motion

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of the body converges (in average) to an uniform one, while for weak body/medium interactions the body with a large initial velocity escapes to infinity increasing its velocity without bound. In the first case we have a reasonable model of viscous friction, in the second case we are in presence of a *runaway particle* (see [27] for a discussion of this effect in the framework of kinetic theories).

Concerning the assumptions on the medium, a natural choice is to consider a system of infinitely many particles, mutually interacting via a pair potential Φ, and interacting with the body via a potential Ψ. Unfortunately, this problem is very hard. Actually, the same definition of the model is not obvious, due to the difficulty to prove the existence of dynamics for infinitely extended systems. We recall that the time evolution of systems with infinitely many particles has been investigated in several papers: we only quote some of the main results [3, 10, 17, 19–21, 24, 25], other references can be found therein. Long time estimates drastically depend on the spatial dimensions of the system, and good enough estimates are known only for one dimensional models with bounded interactions [9]. The runaway effect has been shown to occur for large external force $\boldsymbol{E}$ when the medium is confined in a tube [6], and for any strength of $\boldsymbol{E}$ in the case of particles moving along a line [7]. For singular interactions, the strict one-dimensional case becomes too particular, while in general we can only make conjectures. In [6], for interaction singular as $r^{-\alpha}$ the threshold case is conjectured to be $\alpha = 2$, but a rigorous proof seems too hard.

To make further steps we must simplify the medium. A possible way is to consider a mean field approximation, i.e. to put the body in a Vlasov fluid. In the mean field approximation, well known in Astronomy and in Plasma physics, the background is composed by a gas of particles of mass going to zero and the number of particles per unit volume going to infinity in such a way that the mass density remains finite. The mean field limit of the dynamics of interacting particle systems in the case of finite total mass has been studied in [4, 18, 28, 30] for bounded interaction and in [22] for singular force like $r^{-\alpha}$, $\alpha < 1$. The limit in the case of infinite total mass has been proved in [5] in the case of bounded interaction and spatial dimension one. For unbounded mass, the existence of the dynamics has been proved in three dimensions for a Vlasov fluid with bounded interactions [11], and in two dimension for the Helmholtz equation [12].

In this framework, some results have been recently obtained. In [13, 15] the interaction Ψ is assumed of hard core form, precisely $\Psi(|\boldsymbol{r}|) = \infty$ for $\boldsymbol{r} \in \Lambda$, $\Psi = 0$ otherwise, where Λ is a convex set of $\mathbb{R}^3$ (this means that an element of the medium evolves freely out of Λ and interacts elastically on $\partial\Lambda$). The initial phase space density of the medium is assumed spatially homogeneous out of Λ and with Gaussian distribution of the velocities. It is proved the existence of a stationary state in the motion of the body for any intensity of a spatially constant force $\boldsymbol{E}$, and a detailed analysis of the approach to the limiting velocity is given. This result has been extended to forces not constant in space in [14], and a similar result has been obtained changing the elastic boundary condition with the diffusive ones (see [1, 2] for analytical and numerical results). In the case of singular interaction of the form $r^{-\alpha}$, $\alpha > 0$ the problem is open. We only recall the paper [8], where the runaway particle effect has been proved when Ψ has the form $r^{-\alpha}$, $\alpha < 2$, the medium is composed by a Vlasov fluid, the mutual interaction Φ is bounded, the influence of the body on the fluid motion is neglected, and the system has initially an one-dimensional symmetry.

In the present paper we assume $\Phi = 0$, the body/medium interaction Ψ bounded and with a short range, the background initially at rest, and we find not only the runaway effect but also the exact asymptotic velocity of the body. Then we discuss at a heuristic level the case in which Ψ is always at short range but diverging at short distances as $r^{-\alpha}$, $\alpha < 2$, case in which also the runaway effect is expected to take place. In [6] it has been conjectured that

$\alpha = 2$ is the threshold value for the existence or not of the runaway effect. A discussion on the existence of a stationary state for this model [16] improves this conjecture.

The plan of the paper is the following. In Sect. 2 we introduce the model and state the main result. In Sect. 3 we prove the result for Ψ bounded. Finally, in Sect. 4 we discuss at a heuristic level the case of singular interaction.

2 The Model

We consider the motion of a point body of mass M and position $\boldsymbol{\xi}$, under the action of a constant force $\boldsymbol{E}$ of intensity E and directed along the x_1-axis, i.e. $\boldsymbol{E} = (E, 0, 0)$. The body is immersed in a fluid of free particles, that interact with the body via a force of pair potential $\Psi(|\boldsymbol{r}|)$. We assume the medium to be a *Vlasov fluid* of free particles in three dimensions, namely the pair of functions,

$$(\boldsymbol{x}, \boldsymbol{v}) \to (\boldsymbol{X}(\boldsymbol{x}, \boldsymbol{v}; t), \boldsymbol{V}(\boldsymbol{x}, \boldsymbol{v}; t)), \qquad f_0(\boldsymbol{x}, \boldsymbol{v}) \to f(\boldsymbol{x}, \boldsymbol{v}; t), \tag{2.1}$$

with $(\boldsymbol{x}, \boldsymbol{v}, t) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$, solution to

$$\dot{\boldsymbol{X}}(\boldsymbol{x}, \boldsymbol{v}; t) = \boldsymbol{V}(\boldsymbol{x}, \boldsymbol{v}; t), \tag{2.2}$$

$$\dot{\boldsymbol{V}}(\boldsymbol{x}, \boldsymbol{v}; t) = -\nabla_{\boldsymbol{X}} \Psi(|\boldsymbol{X}(\boldsymbol{x}, \boldsymbol{v}; t) - \boldsymbol{\xi}(t)|), \tag{2.3}$$

$$(\boldsymbol{X}(\boldsymbol{x}, \boldsymbol{v}; 0), \boldsymbol{V}(\boldsymbol{x}, \boldsymbol{v}; 0)) = (\boldsymbol{x}, \boldsymbol{v}), \tag{2.4}$$

$$f(\boldsymbol{X}(\boldsymbol{x}, \boldsymbol{v}; t), \boldsymbol{V}(\boldsymbol{x}, \boldsymbol{v}; t), t) = f_0(\boldsymbol{x}, \boldsymbol{v}), \tag{2.5}$$

where $\boldsymbol{\xi}(t)$ is the coordinate of the point body, f is the mass density in phase space of the free gas, and $(\boldsymbol{X}, \boldsymbol{V})$ are the characteristics of the fluid.

In the case of smooth initial data, $f(\boldsymbol{X}(\boldsymbol{x}, \boldsymbol{v}; t), \boldsymbol{V}(\boldsymbol{x}, \boldsymbol{v}; t), t)$ satisfies the differential equation,

$$(\partial_t + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} - \nabla \Psi(|\boldsymbol{x} - \boldsymbol{\xi}(t)|) \cdot \nabla_{\boldsymbol{v}}) f(\boldsymbol{x}, \boldsymbol{v}; t) = 0. \tag{2.6}$$

The system (2.2)–(2.5) is completed by the equation of motion of the body,

$$M\ddot{\boldsymbol{\xi}}(t) = \boldsymbol{E} - \int \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{v}\, \nabla_{\boldsymbol{\xi}} \Psi(|\boldsymbol{\xi}(t) - \boldsymbol{x}|))\, f(\boldsymbol{x}, \boldsymbol{v}; t), \tag{2.7}$$

with initial conditions

$$\boldsymbol{\xi}(0) = \boldsymbol{\xi}_0, \qquad \dot{\boldsymbol{\xi}}(0) = \dot{\boldsymbol{\xi}}_0. \tag{2.8}$$

This kind of system, in which a Vlasov fluid is coupled to a massive body, has been firstly introduced in connection with the so-called piston problem, see [23] and also [26] and references quoted therein. In our problem the fluid is unbounded and so, in principle, the existence of the solutions is not obvious. In fact, it is easy to exhibit initial conditions for which the time evolution produces singularity in the motion after a finite time. This happens because very far away particles could arrive very fastly close to the body. These situations are pathologic from a physical point of view and can be removed in the case of bounded interaction by assuming that initially

$$f_0(\boldsymbol{x}, \boldsymbol{v}) \le \rho_0 \left(\frac{\beta}{2\pi}\right)^{3/2} \mathrm{e}^{-\beta \boldsymbol{v}^2/2}, \qquad \rho_0 > 0, \tag{2.9}$$

where $\beta = (kT)^{-1} > 0$, T is the temperature, and k the Boltzmann constant, i.e. f is bounded by a homogeneous Gibbs state. In this case the proof of existence and uniqueness of solutions is quite easy. It can be achieved by means of a limiting procedure, where the solution is constructed as the limit of a suitable sequence of partial dynamics of finite total mass.

As discussed at the end of the introduction, in this paper we shall assume that the fluid is initially at rest, with constant spatial density $\rho(\boldsymbol{x}, 0) = \rho_0 > 0$. This corresponds to the limiting case $\beta = \infty$ (zero temperature), which can be defined by means of the Lagrangian formulation,

$$\begin{cases} \ddot{\boldsymbol{X}}(\boldsymbol{x}, \boldsymbol{0}; t) = -\nabla_{\boldsymbol{X}} \Psi(|\boldsymbol{X}(\boldsymbol{x}, \boldsymbol{0}; t) - \boldsymbol{\xi}(t)|), \\ M\ddot{\boldsymbol{\xi}}(t) = \boldsymbol{E} - \rho_0 \displaystyle\int \mathrm{d}\boldsymbol{x}\, \nabla_{\boldsymbol{\xi}} \Psi(|\boldsymbol{\xi}(t) - \boldsymbol{X}(\boldsymbol{x}, \boldsymbol{0}; t)|), \\ (\boldsymbol{X}(\boldsymbol{x}, \boldsymbol{0}; 0), \dot{\boldsymbol{X}}(\boldsymbol{x}, \boldsymbol{0}; 0)) = (\boldsymbol{x}, \boldsymbol{0}), \qquad (\boldsymbol{\xi}(0), \dot{\boldsymbol{\xi}}(0)) = (\boldsymbol{\xi}_0, \dot{\boldsymbol{\xi}}_0). \end{cases} \tag{2.10}$$

Also in this case the existence and uniqueness of global solutions is obtained by means of a limiting procedure. We only sketch the main steps. We first introduce the "n-partial dynamics" $(\boldsymbol{X}^{(n)}(\boldsymbol{x}, \boldsymbol{0}; t), \boldsymbol{\xi}^{(n)}(t))$, governed by a system like (2.10) where $(2.10)_2$ is replaced by

$$M\ddot{\boldsymbol{\xi}}^{(n)}(t) = \boldsymbol{E} - \rho_0 \int_{|\boldsymbol{x}| \le n} \mathrm{d}\boldsymbol{x}\, \nabla_{\boldsymbol{\xi}} \Psi(|\boldsymbol{\xi}^{(n)}(t) - \boldsymbol{X}^{(n)}(\boldsymbol{x}, \boldsymbol{0}; t)|). \tag{2.11}$$

The well-posedness of this truncated problem can be established by a standard argument based on Picard approximations. The solution to the original problem is now obtained in the limit $n \to \infty$. Such a limit exists since the velocities are bounded uniformly with respect to the cut-off parameter n. More precisely, for each $T_0 > 0$ there exists a positive real $C_0 = C_0(\rho_0, \dot{\boldsymbol{\xi}}_0, T_0)$ such that

$$\left|\dot{\boldsymbol{X}}^{(n)}(\boldsymbol{x}, \boldsymbol{0}; t)\right| \le \|\nabla \Psi\|_\infty T_0, \qquad \left|\dot{\boldsymbol{\xi}}^{(n)}(t)\right| \le C_0 \quad \forall t \in [0, T_0].$$

In fact, the first inequality is an immediate consequence of $(2.10)_1$; the second one then follows from (2.11) and the previous one. We omit the details.

3 Bounded Interaction

Assume $\Psi(|\boldsymbol{r}|)$ be a twice differentiable function of $\boldsymbol{r} \in \mathbb{R}^3$ and let exist a positive constant $r_0 < \infty$ such that $\Psi(r) = 0$ if $r > r_0$. We consider the system (2.10) and assume that initially the point body is in the origin, $\boldsymbol{\xi}_0 = (0, 0, 0)$, with velocity along the x_1-axis, $\dot{\boldsymbol{\xi}}_0 = (\dot{\xi}_0, 0, 0)$, $\dot{\xi}_0 > 0$. By symmetry the body will move along the x_1-axis, i.e. $\boldsymbol{\xi}(t) = (\xi(t), 0, 0)$ for some $\xi(t) \in \mathbb{R}$.

Theorem 3.1 *For each intensity E of the force, there exists a threshold value $\dot\xi_*$ such that for $\dot\xi_0 > \dot\xi_*$ the body escapes to infinity with (asymptotically) a uniformly accelerated motion,*

$$\lim_{t\to\infty} \frac{\dot\xi(t)}{t} = \frac{E}{M}. \tag{3.1}$$

We have to prove that the friction force exerted by the fluid on the body is initially bounded and vanishes asymptotically in time. This will be achieved by means of a bootstrap argument. We start with some preliminary notation. We define

$$T^* := \sup\big\{t > 0\colon\ G(s) < \dot\xi(s) < 3G(s)\ \forall\, s \in [0,t]\big\}, \tag{3.2}$$

where

$$G(t) := \frac{1}{2}\left(\dot\xi_0 + \frac{E}{M}t\right). \tag{3.3}$$

Since $\dot\xi_0 > 0$, by the continuity of the motion the set on the right hand side of (3.2) is non empty, thence $T^* \in (0,+\infty]$. We shall study the evolution during the time interval $[0,T^*)$, getting sharper estimates if $\dot\xi_0$ is large enough, that will imply $T^* = +\infty$ and (3.1).

In the next two preliminary lemmata we write $\boldsymbol{x}(t) = (x_1(t), x_2(t), x_3(t))$ as a shorthand notation for a generic characteristic $\boldsymbol{X}(\boldsymbol{x},\boldsymbol{0};t)$ and let

$$\begin{aligned} \tau &:= \inf\{t \in [0,T^*)\colon\ |\boldsymbol{\xi}(t) - \boldsymbol{x}(t)| \le r_0\}, \\ \tau + \delta &:= \sup\{t \in [0,T^*)\colon\ |\boldsymbol{\xi}(t) - \boldsymbol{x}(t)| \le r_0\}, \end{aligned} \tag{3.4}$$

setting $\tau = \tau + \delta = T^*$ if the sets on the right hand side are empty. In other words, if a fluid particle interacts with the body during the time interval $[0,T^*)$ then $\tau < T^*$ denotes the time at which the interaction begins and $\tau + \delta \le T^*$ the time at which it finishes. In Lemma 3.1 below, we show that if the initial velocity of the body is large enough then the time of interaction with a fluid particle is very short and the acceleration of the body is uniformly bounded in time. As a consequence, in Lemma 3.2, we then show that the momentum transferred by the body to a fluid particle is very small.

Lemma 3.1 *There exist $\gamma, A > 0$ such that if $\dot\xi_0 > \gamma$ the following holds. For any characteristic $\boldsymbol{x}(\cdot)$,*

$$\delta \le \frac{5r_0}{G(\tau)}, \tag{3.5}$$

$$|\ddot\xi(t)| \le A \quad \forall t \in [0,T^*). \tag{3.6}$$

Proof We start with the proof of (3.5). Let $s_\tau := 5r_0/G(\tau)$ and consider a characteristic $\boldsymbol{x}(t)$ such that $\tau + s_\tau < T^*$ (otherwise there is nothing to prove). Since $\dot{\boldsymbol{x}}(\tau) = 0$, for any $t \in [\tau, T^*)$,

$$\xi(t) - x_1(t) = \xi(\tau) - x_1(\tau) + \int_\tau^t \mathrm{d}s\, \dot\xi(s) - \int_\tau^t \mathrm{d}s\,(s-\tau)\,\ddot x_1(s).$$

By the equation of motion $(2.10)_1$, $|\ddot{\boldsymbol{x}}(t)| \le \|\Psi'\|_\infty$. Then, by (3.2), (3.3), and (3.4) it follows that, for any $t \in [\tau, T^*)$,

$$\xi(t) - x_1(t) \ge -r_0 + G(\tau)(t-\tau) - \frac{1}{2}\|\Psi'\|_\infty (t-\tau)^2.$$

Since $G(\tau) \ge \dot\xi_0/2$, for $\dot\xi_0 > 4\sqrt{\|\Psi'\|_\infty r_0}$ the right hand side in the above display is larger than r_0 if $t-\tau \in (s_-, s_+)$ with

$$s_- = \frac{4r_0}{G(\tau) + \sqrt{G(\tau)^2 - 4\|\Psi'\|_\infty r_0}}, \qquad s_+ = \frac{G(\tau) + \sqrt{G(\tau)^2 - 4\|\Psi'\|_\infty r_0}}{\|\Psi'\|_\infty}.$$

Clearly $s_\tau \in (s_-, s_+)$ for any $\dot\xi_0$ sufficiently large. Furthermore, $|\dot x_1(\tau + s_\tau)| \le \|\Psi'\|_\infty s_\tau = 5\|\Psi'\|_\infty r_0/G(\tau)$. Therefore, we can find $\gamma > 0$ so large that, for any $\dot\xi_0 > \gamma$,

$$\xi(\tau + s_\tau) - x_1(\tau + s_\tau) > r_0, \qquad \dot x_1(\tau + s_\tau) < G(\tau).$$

Since $\dot\xi(t) > G(\tau)$ for $t \in [\tau, T^*)$, the fluid particle cannot interact again with the body after time s_τ, i.e. $\delta \le s_\tau$ and (3.5) is proved.

The bound (3.6) is now a consequence of (3.5). In fact, since $\dot\xi < 3G$ on $[0, T^*)$ and recalling that the fluid particle is initially at rest, for any $\dot\xi_0 > \gamma$ and $t \in [\tau, \tau+\delta]$,

$$\begin{aligned} |\boldsymbol{\xi}(t) - \boldsymbol{x}(0)| &= |\boldsymbol{\xi}(t) - \boldsymbol{x}(\tau)| \le r_0 + |\boldsymbol{\xi}(t) - \boldsymbol{\xi}(\tau)| && (3.7)\\ &\le r_0 + 3G(\tau+\delta)\delta \le r_0 + 15 r_0 \frac{G(\tau+\delta)}{G(\tau)} \\ &\le r_0 + 15 r_0\left(1 + \frac{E\delta}{M\dot\xi_0}\right) \le r_0 + 15 r_0 \left(1 + \frac{10 r_0 E}{M\dot\xi_0^2}\right) \\ &\le 16 r_0 + \frac{150 r_0^2 E}{M\gamma^2} =: R_\gamma, && (3.8) \end{aligned}$$

where we used (3.5) and definitions (3.3) and (3.4). Therefore, solely the initial positions $\boldsymbol{x}$ in the ball of center $\boldsymbol{\xi}(t)$ and radius R_γ contribute to the integration in the right hand side of $(2.10)_2$. Hence, (3.6) holds with $A = (E/M) + c\|\Psi'\|_\infty$ for a sufficiently large constant c. □

Lemma 3.2 *There exist $\gamma' > \gamma$ and $B, C > 0$ such that if $\dot\xi_0 > \gamma'$ the following holds. For any characteristic $\boldsymbol{x}(\cdot)$,*

$$|\dot{\boldsymbol{x}}(t)| \le \frac{B}{G(\tau)} \quad \forall t \in [0, T^*) \tag{3.9}$$

and, if $\tau + \delta < T^*$,

$$|\dot x_1(t)| = |\dot x_1(\tau+\delta)| \le \frac{C}{G(\tau)^3} \quad \forall t \in [\tau+\delta, T^*). \tag{3.10}$$

Proof Actually, the bound (3.9) is a consequence of (3.5) and it is therefore valid for any $\dot\xi_0 > \gamma$. Indeed,

$$|\dot{\boldsymbol{x}}(t)| \le \int_\tau^{\tau+\delta} \mathrm{d}s \left|\Psi'(|\boldsymbol{x}(s) - \boldsymbol{\xi}(s)|)\right| \le \frac{B}{G(\tau)} \quad \forall t \in [0, T^*), \tag{3.11}$$

where $B := 5 r_0 \|\Psi'\|_\infty$. Assume now that $\tau + \delta < T^*$. We expect that the total variation of $\dot x_1$ is really very small for large velocities, since the effects produced when $x_1 > \xi$ compensate in part those produced when $x_1 < \xi$. To show this fact we introduce the quantity $\boldsymbol{p}(t) = (p_1(t), p_2(t), p_3(t))$ defined as

$$\boldsymbol{p}(t) = \dot{\boldsymbol{x}}(t) + \frac{\Psi(|\boldsymbol{x}(t) - \boldsymbol{\xi}(t)|)}{|\dot{\boldsymbol{x}}(t) - \dot{\boldsymbol{\xi}}(t)|^2}(\dot{\boldsymbol{x}}(t) - \dot{\boldsymbol{\xi}}(t)). \tag{3.12}$$

Obviously,

$$\boldsymbol{p}(t) = \dot{\boldsymbol{x}}(t) \quad \forall\, t \in [0,\tau) \cup (\tau+\delta, T^*),$$

so that, since before the collision the fluid particle is at rest,

$$|\dot{x}_1(\tau+\delta)| \le \int_\tau^{\tau+\delta} \mathrm{d}s\, |\dot{p}_1(s)|. \tag{3.13}$$

We next show that during the collision $\dot{p}_1(t)$ is quite small, thence a finer estimate for $|\dot{x}_1(\tau+\delta)|$ will follow by (3.13). By the equation of motion $(2.10)_1$,

$$\ddot{\boldsymbol{x}}(t) = -\frac{\Psi'(|\boldsymbol{x}(t)-\boldsymbol{\xi}(t)|)}{|\boldsymbol{x}(t)-\boldsymbol{\xi}(t)|}\big[\boldsymbol{x}(t)-\boldsymbol{\xi}(t)\big]. \tag{3.14}$$

By (3.12), (3.14), and some straightforward computation,

$$\begin{aligned}\dot{p}_1(t) = &-\frac{\Psi'(|\boldsymbol{x}-\boldsymbol{\xi}|)}{|\boldsymbol{x}-\boldsymbol{\xi}||\dot{\boldsymbol{x}}-\dot{\boldsymbol{\xi}}|^2}\big[(x_1-\xi)(\dot{x}_2^2+\dot{x}_3^2)+(\dot{\xi}-\dot{x}_1)(x_2\dot{x}_2+x_3\dot{x}_3)\big]\\ &+\frac{\Psi(|\boldsymbol{x}-\boldsymbol{\xi}|)}{|\dot{\boldsymbol{x}}-\dot{\boldsymbol{\xi}}|^2}\left[-(\ddot{\xi}-\ddot{x}_1)+\frac{2(\dot{\xi}-\dot{x}_1)(\dot{\boldsymbol{\xi}}-\dot{\boldsymbol{x}})\cdot(\ddot{\boldsymbol{\xi}}-\ddot{\boldsymbol{x}})}{|\dot{\boldsymbol{x}}-\dot{\boldsymbol{\xi}}|^2}\right],\end{aligned}$$

whence

$$\begin{aligned}|\dot{p}_1(t)| &\le \big|\Psi'(|\boldsymbol{x}-\boldsymbol{\xi}|)\big|\,\frac{|\dot{\boldsymbol{x}}|^2+2|\dot{\boldsymbol{x}}|}{|\dot{\boldsymbol{x}}-\dot{\boldsymbol{\xi}}|^2}+3\|\Psi\|_\infty\frac{|\ddot{\xi}|+|\ddot{\boldsymbol{x}}|}{|\dot{\boldsymbol{x}}-\dot{\boldsymbol{\xi}}|^2}\\ &\le \big|\Psi'(|\boldsymbol{x}-\boldsymbol{\xi}|)\big|\,\frac{|\dot{\boldsymbol{x}}|^2+2|\dot{\boldsymbol{x}}|+3\|\Psi\|_\infty}{|\dot{\boldsymbol{x}}-\dot{\boldsymbol{\xi}}|^2}+\frac{3A\|\Psi\|_\infty}{|\dot{\boldsymbol{x}}-\dot{\boldsymbol{\xi}}|^2},\end{aligned}$$

where we used (3.6) and again (3.14) in the last inequality. Since $\dot{\xi}(t) > G(\tau) \ge \dot{\xi}_0/2$ on $[\tau, T^*)$, by (3.9) and the above estimate we can find $\gamma' > \gamma$ so large that if $\dot{\xi}_0 > \gamma'$ then

$$|\dot{p}_1(t)| \le \frac{B+4\|\Psi\|_\infty}{G(\tau)^2}\big|\Psi'(|\boldsymbol{x}(t)-\boldsymbol{\xi}(t)|)\big| + \frac{4A\|\Psi\|_\infty}{G(\tau)^2} \quad \forall\, t \in [0,T^*).$$

By plugging the previous inequality in (3.13), estimating the time integral of the force as in (3.11), and using (3.5) we get

$$|\dot{x}_1(\tau+\delta)| \le \frac{C}{G(\tau)^3},$$

where $C := B^2 + 4B\|\Psi\|_\infty + 20A\|\Psi\|_\infty r_0$. Since $\dot{x}_1(t) = \dot{x}_1(\tau+\delta)$ for any $t \in [\tau+\delta, T^*)$, the estimate (3.10) is thus proved. □

Proof of Theorem 3.1 We shall prove the theorem with a large enough $\dot{\xi}_* > \gamma'$, γ' as in Lemma 3.2. By the equations of motion $(2.10)_1$ and $(2.10)_2$, since $\dot{\boldsymbol{\xi}}(t) = (\dot{\xi}(t), 0, 0)$ and denoting by $X_1(\boldsymbol{x}, \boldsymbol{0}; t)$ the first component of the characteristic $\boldsymbol{X}(\boldsymbol{x}, \boldsymbol{0}; t)$, we have

$$M\ddot{\xi}(t) + \rho_0 \int \mathrm{d}\boldsymbol{x}\, \ddot{X}_1(\boldsymbol{x}, \boldsymbol{0}; t) - E = 0.$$

Therefore, by integrating in time,

$$M\dot{\xi}(t) + \rho_0 \int \mathrm{d}\boldsymbol{x}\, \dot{X}_1(\boldsymbol{x}, \boldsymbol{0}; t) - Et = M\dot{\xi}_0,$$

whence

$$\left|\dot{\xi}(t) - \dot{\xi}_0 - \frac{E}{M}t\right| \le \frac{\rho_0}{M} \int \mathrm{d}\boldsymbol{x} \left|\dot{X}_1(\boldsymbol{x}, \boldsymbol{0}; t)\right|. \tag{3.15}$$

Since $\xi(\cdot)$ is increasing on $[0, T^*)$, if $\dot{X}_1(\boldsymbol{x}, \boldsymbol{0}; t) \neq 0$ for some $t \in [0, T^*)$ then the initial position $\boldsymbol{x} = (x_1, x_2, x_3) =: (x_1, \boldsymbol{x}^\perp)$ of the fluid particle is such that $x_1 \in (-r_0, \xi(t) + r_0)$ and $|\boldsymbol{x}^\perp| \le r_0$. Moreover, by (3.7), if $t \in [\tau, \tau + \delta]$ then $|x_1 - \xi(t)| \le R_\gamma$. Therefore,

$$\begin{aligned}\int \mathrm{d}\boldsymbol{x} \left|\dot{X}_1(\boldsymbol{x}, \boldsymbol{0}; t)\right| &= \int_{-r_0}^{N_0} \mathrm{d}x_1 \int_{|\boldsymbol{x}^\perp| \le r_0} \mathrm{d}\boldsymbol{x}^\perp \left|\dot{X}_1(\boldsymbol{x}, \boldsymbol{0}; t)\right| \\ &\quad + \sum_{k=N_0}^{N_t - 1} \int_k^{k+1} \mathrm{d}x_1 \int_{|\boldsymbol{x}^\perp| \le r_0} \mathrm{d}\boldsymbol{x}^\perp \left|\dot{X}_1(\boldsymbol{x}, \boldsymbol{0}; \tau + \delta)\right| \\ &\quad + \int_{N_t}^{\xi(t) + r_0} \mathrm{d}x_1 \int_{|\boldsymbol{x}^\perp| \le r_0} \mathrm{d}\boldsymbol{x}^\perp \left|\dot{X}_1(\boldsymbol{x}, \boldsymbol{0}; t)\right|,\end{aligned}$$

with $N_0 := [r_0] + 1$ and $N_t := [\xi(t) - R_\gamma] - 1$. The quantities $|\dot{X}_1(\boldsymbol{x}, \boldsymbol{0}; t)|$ and $|\dot{X}_1(\boldsymbol{x}, \boldsymbol{0}; \tau + \delta)|$ can be bounded, respectively, as in (3.9) and (3.10). Moreover, since $\dot{\xi} < 3G$ on $[0, T^*)$, if $x_1 \in [k, k+1]$ then $3G(\tau)\tau + r_0 \ge k$, so that, by (3.3),

$$\frac{6M}{E} G(\tau)^2 - \frac{3M\dot{\xi}_0}{E} G(\tau) - (k - r_0) \ge 0 \quad \forall x_1 \in [k, k+1],$$

thence

$$G(\tau) \ge \sqrt{\frac{E}{6M}(k - r_0)} \quad \forall k > N_0 \; \forall x_1 \in [k, k+1].$$

Therefore,

$$\int \mathrm{d}\boldsymbol{x} \left|\dot{X}_1(\boldsymbol{x}, \boldsymbol{0}; t)\right| \le B\, \frac{(2N_0 + R_\gamma)\pi r_0^2}{G(0)} + C \sum_{k=N_0+1}^{\infty} \left(\frac{6M}{E(k - r_0)}\right)^{3/2}.$$

In conclusion, there exists $C_1 > 0$ such that for any $\dot{\xi}_0 > \gamma'$ and $t \in [0, T^*)$,

$$\left|\dot{\xi}(t) - \dot{\xi}_0 - \frac{E}{M}t\right| \le C_1. \tag{3.16}$$

Recalling the definition (3.3) of G, by (3.16) it follows that there exists $\dot{\xi}_* > \gamma'$ such that

$$\frac{3}{2} G(t) < \dot{\xi}(t) < \frac{5}{2} G(t) \quad \forall t \in [0, T^*) \; \forall \dot{\xi}_0 > \dot{\xi}_*.$$

By the continuity of the motion and the definition (3.2) of T^* the above bound implies that $T^* = +\infty$ for any $\dot{\xi}_0 > \dot{\xi}_*$. In particular, (3.16) is valid for any positive time and the limit (3.1) is thus verified for $\dot{\xi}_0 > \dot{\xi}_*$. The theorem is proved. □

Remark 3.1 We have assumed the density of the fluid constant in the whole space. Of course, the result and the proof do not change if we assume the initial density $\rho_0(\boldsymbol{x})$ with an axial symmetry around the x_1-axis and $\rho_0(\boldsymbol{x}) \to$ const. fast enough as $|\boldsymbol{x}| \to \infty$. Indeed, this only changes the transient evolution, but the asymptotic behavior of the motion remains unchanged.

It is natural to ask what happens if we assume the fluid initially distributed as a Gibbs state like (2.9) (Theorem 3.1 here corresponds to the case $\beta = \infty$). For E larger than $\|\Psi'\|_\infty$ the result is still true [29], while for smaller E it depends on the sign of $\Psi'(r)$: we conjecture that if $\Psi'(r) \le 0$ the result remains valid; while, if $\Psi'(r) > 0$ for some r, the quantity $\dot{\xi}(t)/t$ should converge to some constant possibly smaller than E/M. In fact, the body could capture a small part of fluid, which remains with it forever, with the only effect to increase the effective inertia (i.e. mass) of the body.

4 Singular Interaction

The spirit of this section is different from the previous one, where we have given a mathematically rigorous proof on the asymptotic behavior of the system. Here we confine ourself to a heuristic discussion of the main issues in a possible proof.

We assume $\Psi(r)$ be a twice differentiable function for $r > 0$ and there exist two positive constants $r_1, r_0 < \infty$ such that $\Psi = gr^{-\alpha}, g > 0, 0 < \alpha < 2$, if $r < r_1$, Ψ equals to a not increasing function if $r_1 \le r \le r_0$, $\Psi = 0$ if $r > r_0$. Initially we put the point body in the origin with a fix velocity $\dot{\xi}(0) = \dot{\xi}_0 > 0$, and the fluid at rest with a constant density $\rho(\boldsymbol{x}, 0) = \rho_0 > 0$. We discuss the following conjecture, analogous to Theorem 3.1.

Conjecture *For each intensity E of the force*, *there exists a threshold value $\dot{\xi}_*$ such that for $\dot{\xi}_0 > \dot{\xi}_*$ the body escapes to infinity with* (*asymptotically*) *a uniformly accelerated motion*,

$$\lim_{t\to\infty} \frac{\dot{\xi}(t)}{t} = \frac{E}{M}. \tag{4.1}$$

The strategy of a possible proof is similar to that used in the bounded case: we assume a behavior of the body and we evaluate the viscous friction produced by the fluid. Using an adiabatic invariant we show that the friction is very small and vanishes at long times, so that the behavior of the body is better than the assumption. Unfortunately, we do not prove some (minor) steps and the discussion remains only heuristic.

In the present analysis two new points arises: (a) the interaction is singular and so the particles of fluid near the x_1-axis drastically change their velocity whatever large is the velocity of the body; (b) some particles of the fluid can collide infinitely many times. The first difficulty is solved by proving that these *bad* particles are very few when the velocity of the body becomes large; the second difficulty is solved by showing the effect of recollisions is negligible.

As in the bounded case we analyze the motion in the time interval $[0, T^*)$ where T^* is defined as in (3.2). It is reasonable to believe that also in this case the analogous of Lemma 3.1 remains valid, but we did not have proved it rigorously. However, physically it is quite obvious. In particular, since the body feels the average force due to the fluid particles in a neighborhood of it, the bound (3.6) should be valid with a better constant A, depending on some smoother norm of the potential rather than the maximum of the force. Moreover, at the end of the discussion it will be clear that a proof using this assumption should imply that the acceleration goes to zero in average on a time interval of length vanishing as $t \to \infty$. But a rigorous proof of (3.6) in this context appears cumbersome and not trivial.

We explicitly discuss the more difficult case $1 < \alpha < 2$, starting with the study of the first collision.

We analyze the motion of a fluid particle. We denote by η its impact parameter, that is the distance between the incoming fluid particle and the x_1-axis. In the plane (x_2, x_3) we consider the disks D_k, centered in the origin and of radius $\eta_k = 2^k \eta_0$, $k = 0, 1, 2, \ldots$. We choose $\eta_0 = \dot{\xi}(t)^{-(1+\epsilon)}$, $\epsilon > 0$, so that the particles with $\eta \le \eta_0$ contribute at most as Const. $\dot{\xi}(t)^{-2\epsilon}$ in the momentum transferred to the body (per unit time). In fact, the area of this disk is $\pi \eta_0^2$, the body meets per unit time a number of fluid particles proportional to $\dot{\xi}(t)$, in each collision it looses a momentum $2\dot{\xi}(t)$ (corresponding to an elastic collision with a disk) and so that the lost momentum due to particle hitting in this disk is at most Const. $\dot{\xi}^2(t)\eta_0^2 =$ Const. $\dot{\xi}(t)^{-2\epsilon}$.

We now evaluate the effect of the collisions in the k-annulus $D_k \setminus D_{k-1}$. First we remark that, by arguing as at the beginning of the previous section, it is possible to show that the body has only one collision with each fluid particle posed in a k-annulus, and then it overcomes such particles. Hence, the momentum lost during this collision can be bounded as in (3.10), but here a better estimate on the constant C is needed to conclude the argument. We recall that in Lemma 3.2 the estimate (3.10) is proved with $C = B^2 + 4B\|\Psi\|_\infty + 20A\|\Psi\|_\infty r_0$, where $B = 5r_0\|\Psi'\|_\infty$ comes out from the upper bound of the time integral of the force appearing in (3.11). In this case, since the force is repulsive and the fluid particle is initially at rest, during the collision its distance from the x_1-axes increases. This implies that the norm $\|\Psi\|_\infty$ appearing in the expression of C can be replaced by Const. $\eta_k^{-\alpha}$ and that

$$\int_\tau^{\tau+\delta} \mathrm{d}s \left|\Psi'(|\boldsymbol{x}(s) - \boldsymbol{\xi}(s)|)\right| \le \int_\tau^{\tau+\delta} \mathrm{d}s \left|\Psi'\Big(\sqrt{(x_1(s) - \xi(s))^2 + \eta_k^2}\Big)\right|.$$

To evaluate the integral on the right hand side, we recall that for $\dot{\xi}_0$ large enough we have e.g. $\dot{\xi} - \dot{x}_1 \ge G(\tau)/2$ during the collision. Therefore, by the change of integration variable $s \to u := \xi - x_1$,

$$\begin{aligned}\int_\tau^{\tau+\delta} \mathrm{d}s \left|\Psi'(|\boldsymbol{x}(s) - \boldsymbol{\xi}(s)|)\right| &\le \frac{2}{G(\tau)} \int \mathrm{d}u \left|\Psi'\Big(\sqrt{u^2 + \eta_k^2}\Big)\right| \\ &= \frac{4\eta_k}{G(\tau)} \int_1^\infty \mathrm{d}\sigma\, \frac{\sigma}{\sqrt{\sigma^2 - 1}} \left|\Psi'(\eta_k \sigma)\right| \le \text{Const.}\ \frac{1}{G(\tau)\eta_k^\alpha},\end{aligned}$$

where in the last inequality we used the explicit form of $\Psi(r)$ as stated at the beginning of the section. Therefore, also the constant B appearing in the expression of C can be replaced by Const. $\eta_k^{-\alpha}$. The momentum transferred by the body to the fluid particle after a collision is thus estimated by Const. $\eta_k^{-2\alpha}\dot{\xi}(t)^{-3}$ (here we confuse $G(\tau)$ with $\dot{\xi}(t)$ for a collision occurring around time t). The area A_k of the k-annulus is Const. η_k^2 and the number of collisions per unit time is of the order of $\dot{\xi}(t)$. In conclusion, the contribution I to the momentum lost due to the fluid particles posed in all the annuli is bounded as

$$I \le \text{Const.}\ \dot{\xi}(t)^{[(2\alpha-2)(1+\epsilon)-2]} \sum_{k=1}^{k^*} 2^{2k(-\alpha+1)}, \qquad \eta_{k^*} \ge r_0. \tag{4.2}$$

If we choose $\epsilon < (2/\alpha) - 1$ we obtain

$$I \le \text{Const.}\, \dot{\xi}(t)^{-2\epsilon}, \tag{4.3}$$

that is a bound analogous to the one obtained for the first disk.

We now roughly discuss the effect of multiple collisions. For the moment, we suppose that the body is governed by a uniformly accelerated motion. The only possibility for a fluid particle to hit the body many times is to be quite close to the x_1-axis. In particular, we start by evaluating how much close for hitting the body twice. Denote by σ the impact parameter of the fluid particle and let t_1 be the time at which the first collision occurs. Due to the monotonicity of the potential, during the collision the fluid particle feels a repulsive force from the x_1-axis which varies its orthogonal velocity by a quantity

$$\delta v^{\perp} \simeq \int_{t_1}^{t_1+\delta} \mathrm{d}s\, |\nabla \Psi(\boldsymbol{x}(s) - \boldsymbol{\xi}(s))|\, \frac{\sigma}{r_0}. \tag{4.4}$$

By estimating the integral as before we obtain

$$\delta v^{\perp} \gtrsim \text{Const.}\, \frac{\sigma}{r_0} \dot{\xi}(t_1)^{-1} r_m^{-\alpha}, \tag{4.5}$$

where r_m is the minimal distance between the fluid particle and the body. Since $r_m \approx \text{Const.}\, \dot{\xi}^{-2/\alpha}$ we conclude that

$$\delta v^{\perp} \gtrsim \text{Const.}\, \sigma \dot{\xi}(t_1). \tag{4.6}$$

After the collision, the fluid particle has gained a velocity which is double than the speed of the body. But the particle then preserves its velocity while the body accelerates almost uniformly, and so it can be reached again after a time of the order of $\dot{\xi}(t_1)$. In conclusion, the fluid particle can hit the body twice if its impact parameter σ is very small,

$$\sigma \lesssim \text{Const.}\, \dot{\xi}(t_1)^{-2}. \tag{4.7}$$

Actually, we are interested in the case when the fluid particle has a third collision. For this to happen the impact parameter has to be very small, precisely

$$\sigma \lesssim C \dot{\xi}(t_1)^{-2} \dot{\xi}(t_2)^{-2} \le \text{Const.}\, \dot{\xi}(t_1)^{-4} \tag{4.8}$$

where t_2 is the time of the second collision and we used $\dot{\xi}(t_2) \ge \dot{\xi}(t_1)$.

To apply these ideas to the true case, some modifications are needed: taking into account that the motion of the body is not exactly uniformly accelerated, but with a small perturbation; moreover, to evaluate the effect of multiple collisions at a fixed time, we should go back in the past up to the initial time; finally, for the repulsive force we need a lower bound (not an approximation as in previous discussion). We do not perform this analysis in detail, but we hope to have convinced the reader that the effects of multiple collisions are negligible.

The conclusion of the proof is now as in the bounded case: for any $\dot{\xi}_0$ large enough and $t \in [0, T^*)$ we obtain

$$\left| \dot{\xi}(t) - \dot{\xi}_0 - \frac{E}{M} t \right| \le \text{Const.},$$

which implies $T^* = +\infty$ and then (4.1).

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References

1. Aoki, K., Cavallaro, G., Marchioro, C., Pulvirenti, M.: On the motion of a body in thermal equilibrium immersed in a perfect gas. M2AN Math. Model. Numer. Anal. **42**, 263–275 (2008)
2. Aoki, K., Tsuji, T., Cavallaro, G.: Approach to the steady motion of a plate moving in a free-molecular gas under a constant external force. Phys. Rev. E **80**, 16309 (2009)
3. Bahn, C., Park, Y.M., Yoo, H.J.: Non equilibrium dynamics of infinite particle systems with infinite range interaction. J. Math. Phys. **40**, 4337–4358 (1999)
4. Braun, W., Hepp, K.: The Vlasov dynamics and its fluctuations in the 1/N limit of interacting classical particles. Commun. Math. Phys. **56**, 101–111 (1977)
5. Buttà, P., Caglioti, E., Marchioro, C.: On the long time behavior of infinitely extended systems of particles interacting via Kac potentials. J. Stat. Phys. **108**, 317–339 (2002)
6. Buttà, P., Caglioti, E., Marchioro, C.: On the motion of a charged particle interacting with an infinitely extended system. Commun. Math. Phys. **233**, 545–569 (2003)
7. Buttà, P., Caglioti, E., Marchioro, C.: On the violation of Ohm's law for bounded interactions: a one dimensional system. Commun. Math. Phys. **249**, 353–382 (2004)
8. Buttà, P., Manzo, F., Marchioro, C.: A simple Hamiltonian model of runaway particle with singular interaction. Math. Models Methods Appl. Sci. **15**, 753–766 (2005)
9. Caglioti, E., Marchioro, M.: On the long time behavior of a particle in an infinitely extended system in one dimension. J. Stat. Phys. **106**, 663–680 (2002)
10. Caglioti, E., Marchioro, C., Pulvirenti, M.: Non-equilibrium dynamics of three-dimensional infinite particle systems. Commun. Math. Phys. **215**, 25–43 (2000)
11. Caglioti, E., Caprino, S., Marchioro, C., Pulvirenti, M.: The Vlasov equation with infinite mass. Arch. Ration. Mech. Anal. **159**, 85–108 (2001)
12. Caprino, S., Marchioro, C., Pulvirenti, M.: On the Vlasov-Helmholtz equations with infinite mass. Commun. Partial Differ. Equ. Phys. **27**, 791–808 (2002)
13. Caprino, S., Marchioro, C., Pulvirenti, M.: Approach to equilibrium in a microscopic model of friction. Commun. Math. Phys. **264**, 167–189 (2006)
14. Caprino, S., Cavallaro, G., Marchioro, C.: On a microscopic model of viscous friction. Math. Models Methods Appl. Sci. **17**, 1369–1403 (2007)
15. Cavallaro, G.: On the motion of a convex body interacting with a perfect gas in the mean-field approximation. Rend. Math. Appl. **27**, 123–145 (2007)
16. Cavallaro, G., Marchioro, C.: Necessary condition on the interaction to obtain a microscopic model of viscous friction. Preprint (2009)
17. Cavallaro, G., Marchioro, C., Spitoni, C.: Dynamics of infinitely many particles mutually interacting in three dimensions via a bounded superstable long-range potential. J. Stat. Phys. **120**, 367–416 (2005)
18. Dobrushin, R.L.: Vlasov equations. Soviet J. Funct. Anal. **13**, 60–110 (1979)
19. Dobrushin, R.L., Fritz, J.: Non equilibrium dynamics of one-dimensional infinite particle system with hard-core interaction. Commun. Math. Phys. **55**, 275–292 (1977)
20. Fritz, J.: Some remarks of non-equilibrium dynamics of infinite particle systems. J. Stat. Phys. **34**, 67539–67556 (1977)
21. Fritz, J., Dobrushin, R.L.: Non-equilibrium dynamics of two-dimensional infinite particle systems with a singular interaction. Commun. Math. Phys. **57**, 67–81 (1977)
22. Hauray, M., Jabin, P.E.: N-particles approximation of the Vlasov-Poisson equation with singular potentials. Arch. Ration. Mech. Anal. **183**, 489–524 (2007)
23. Gruber, C., Piasecki, J.: Stationary motion of the adiabatic piston. Physica A **268**, 412–423 (1999)
24. Lanford, O.E.: Classical mechanics of one-dimensional systems with infinitely many particles. I. An existence theorem. Commun. Math. Phys. **9**, 176–191 (1968)
25. Lanford, O.E.: Classical mechanics of one-dimensional systems with infinitely many particles. II. Kinetic theory. Commun. Math. Phys. **11**, 257–292 (1969)
26. Lebowitz, J.L., Piasecki, J., Sinai, Ya.G.: Scaling dynamics of a massive piston in a ideal gas. In: Hard Ball Systems and Lorentz Gas. Encycl. Math. Sci., vol. 101, pp. 217–227. Springer, Berlin (2000)
27. Landau, L.D., Lifshitz, E.M.: Physical Kinetics: Course in Theoretical Physics, vol. 10. Pergamon Press, London (1981)

28. Neunzert, H.: An introduction to the nonlinear Boltzmann-Vlasov equation. In: Lecture Notes in Mathematics, vol. 1048, pp. 115–123. Springer, Berlin (1981)
29. Sganga, D.: Moto di una particella carica in un fluido di Vlasov libero. Graduate thesis, SAPIENZA Università di Roma (2008) [in Italian]
30. Spohn, H.: On the Vlasov hierarchy. Math. Methods Appl. Sci. **3**, 445–455 (1981)